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Gröbner bases with respect to several orderings and multivariable dimension polynomials

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Abstract

Let $D = K[X]$ be a ring of Ore polynomials over a field K and let a partition of the set of indeterminates into p disjoint subsets be fixed. Considering D as a filtered ring with the natural p-dimensional filtration, we introduce a special type of reduction in a free D-module and develop the corresponding Gröbner basis technique (in particular, we obtain a generalization of the Buchberger Algorithm). Using such a modification of the Gröbner basis method, we prove the existence of a Hilbert-type dimension polynomial in p variables associated with a finitely generated filtered D-module, give a method of computation and describe invariants of such a polynomial. The results obtained are applied in differential algebra where the classical theorems on differential dimension polynomials are generalized to the case of differential structures with several basic sets of derivation operators.

Keywords: Ore polynomials; Differential ring; Differential module; Differential field extension; Gröbner basis; Dimension polynomial

1. Introduction

It is well known that the classical theory of Gröbner bases gives efficient methods of computation of Hilbert polynomials of graded and filtered modules over polynomial rings. Similarly, the theory of Gröbner bases of modules over rings of differential operators developed in Oaku and Shimoyama (1994), Insa and Pauer (1998), Kondrateva et al. (1999, Ch. 4) and some other works provides methods of computation of dimension polynomials of differential modules, differential field extensions, and systems of algebraic differential extensions

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(see Kondrateva et al. (1985), Levin and Mikhalev (1987), Levin and Mikhalev (1991), Mikhalev and Pankratev (1980), and Kondrateva et al. (1999, Ch. 5,9)).

The important role of differential dimension polynomials is determined by at least three factors. First, a dimension polynomial associated with a system of algebraic differential equations expresses the strength of such a system in the sense of A. Einstein. (The significant role of this characteristic in the theory of equations of mathematical physics is described, for example, in Einstein (1953)). Second, a differential dimension polynomial carries certain differential birational invariants, that is, numbers that do not change when we switch to another system of generators of a differential field extension, and therefore characterize the extension itself (Kolchin (1973, Ch. 2); Johnson (1969a); Johnson and Sit (1978); Sit (1978); Mikhalev and Pankratev (1980)). Finally, properties of differential dimension polynomials associated with prime differential ideals provide a powerful tool in the dimension theory of differential rings (Johnson (1969b); Kondrateva et al. (1999, Ch. 7)).

In this paper we generalize the Gröbner basis method to the case of free modules over Ore polynomial rings with several term orderings associated with a partition of the set of variables. As a consequence we obtain essential generalizations of theorems on dimension polynomials of differential and difference modules proved in Johnson (1969a) and Levin (1978). In particular, we find new invariants of such dimension polynomials. Furthermore, using the differential structure of a module of differentials associated with a differential field extension we prove a generalization of the classical Kolchin theorem on differential dimension polynomial (see Kolchin (1964)) and determine new invariants of differential field extensions.

In the preliminary section we introduce basic notation and consider properties of numerical polynomials in several variables, i.e., polynomials that take integer values for all sufficiently large integer values of the arguments. We also recall basic concepts of differential algebra and formulate the Kolchin's theorem. In the next section we consider a ring of Ore polynomials $D = K[X]$, where K is a field of zero characteristic and the set of indeterminates X is a union of p disjoint sets $X_1, \ldots, X_p$, and define a reduction in a finitely generated free module $E = \bigoplus_{i=1}^{m} De_i$ over D that involves p natural term orders associated with the partition of X. We develop the Gröbner basis method for this reduction and, in particular, prove a generalization of the Buchberger algorithm. In Section 4 we use our new technique to prove the existence and obtain a method of computation of a numerical polynomial in p variables that describes the dimensions of components of the natural p-dimensional filtration of a finitely generated D-module. Applying this result we prove a theorem on multivariable differential dimension polynomial and find new differential birational invariants of a finitely generated extension of a differential field.

2. Preliminaries

Throughout the paper **Z**, **N** and **Q** denote the sets of integers, non-negative integers and rational numbers, respectively. By a ring we always mean an associative ring with unity. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring. Unless otherwise indicated, by the module over a ring A we mean a unitary left A-module.

Let K be a field and $\Delta = \{\delta_1, \ldots, \delta_n\}$, $\sigma = \{\alpha_1, \ldots, \alpha_n\}$ sets of skew derivations and injective endomorphisms of the field K, respectively, such that $\delta_i(ab) = \alpha_i(a)\delta_i(b) + \delta_i(a)b$ for every $a, b \in K (1 \leq i \leq n)$ and any two mappings from the set $\Delta \bigcup \sigma$ commute with each other. Let $\Theta = \Theta(X)$ be a free commutative semigroup generated by a set $X = \{x_1, \ldots, x_n\}$, and for any element $\theta = x_1^{k_1} \dots x_n^{k_n} \in \Theta$, let τ_θ denote the endomorphism $\alpha_1^{k_1} \dots \alpha_n^{k_n}$ of the field K. Furthermore, let D denote the vector K-space with the basis Θ (elements of D are of the form $\sum_{\theta\in\Theta} a_\theta\theta$ where $a_\theta \in K$ and only finitely many coefficients a_θ are different from zero). Then D can be treated as a ring if one introduces the multiplication in D according to the rule $x_i a = \alpha_i(a)x_i + \delta_i(a)$ $(a \in K, 1 \le i \le n)$ and the distributive laws. (K and Θ are naturally considered as subsets of D, the products ab and $\theta_1\theta_2$ $(a, b \in K; \theta_1, \theta_2 \in \Theta)$ are defined as the corresponding products in K and Θ, respectively.)

The ring D is called a *ring of Ore polynomials* in $x_1, \dots, x_n$ over K, it will be denoted by $K[x_1, \dots, x_n]$ or $K[X]$. (The fact that this is the common notation for a regular polynomial ring leads to no confusions.)

Examples 2.1. 1. If α_i $(1 \le i \le n)$ are identity mappings, then D is the ring of differential operators over the differential field K with the basic set of derivation operators Δ.
2. If δ_i $(1 \le i \le n)$ are zero derivations of K, then D is the ring of difference operators over the difference field K with the basic set σ. If, in addition, all α_i are identity automorphisms, then D is the usual polynomial ring in variables $x_1, \dots, x_n$ over K.
3. If there is k, $1 \le k < n$, such that $\delta_{k+1} = \dots = \delta_n = 0$ and $\alpha_1, \dots, \alpha_k$ are identity automorphisms of K, then D is the ring of difference–differential operators over the difference–differential field K with the basic derivations $\delta_1, \dots, \delta_k$ and basic translations $\alpha_{k+1}, \dots, \alpha_n$.

Let us fix a partition of the set X into p disjoint subsets:

$$X = X_1 \bigcup \dots \bigcup X_p \tag{2.1}$$

where

$$X_1 = \{x_1, \dots, x_{n_1}\}, X_2 = \{x_{n_1+1}, \dots, x_{n_1+n_2}\}, \dots,$$
$$X_p = \{x_{n_1+\dots n_{p-1}+1}, \dots, x_n\} \ (n_1 + \dots + n_p = n).$$

In what follows, elements of the set Θ will be called *monomials*. By the *order* of a monomial $\theta = x_1^{k_1} \dots x_n^{k_n} \in \Theta$ we mean the number $\operatorname{ord}\theta = \sum_{\nu=1}^{n} k_\nu$, the order of θ with respect to X_i $(1 \le i \le p)$ is defined as $\operatorname{ord}_i\theta = \sum_{\nu=n_1+\dots+n_{i-1}+1}^{n_1+\dots+n_i} k_\nu$ (if $i = 1$, the last sum is replaced by $k_1 + \dots + k_{n_1}$).

Furthermore, we consider p orderings $<_1, \dots, <_p$ of the semigroup Θ defined as follows: $\theta = x_1^{k_1} \dots x_n^{k_n} <_i \theta' = x_1^{l_1} \dots x_n^{l_n}$ if and only if the vector $(\operatorname{ord}_i\theta, \operatorname{ord}\theta, \operatorname{ord}_1\theta, \dots, \operatorname{ord}_{i-1}\theta, \operatorname{ord}_{i+1}\theta, \dots, \operatorname{ord}_p\theta, k_{n_1+\dots+n_{i-1}+1}, \dots, k_{n_1+\dots+n_i}, k_1, \dots, k_{n_1+\dots+n_{i-1}}, k_{n_1+\dots+n_i+1}, \dots, k_n)$ is less than the vector $(\operatorname{ord}_i\theta', \operatorname{ord}\theta', \operatorname{ord}_1\theta', \dots, \operatorname{ord}_{i-1}\theta', \operatorname{ord}_{i+1}\theta', \dots, \operatorname{ord}_p\theta', l_{n_1+\dots+n_{i-1}+1}, \dots, l_{n_1+\dots+n_i}, l_1, \dots, l_{n_1+\dots+n_{i-1}}, l_{n_1+\dots+n_i+1}, \dots, l_n)$ with respect to the lexicographic order on $\mathbf{N}^{n+p+1}$.

For any $r_1, \dots, r_p \in \mathbf{N}$, let $\Theta(r_1, \dots, r_p) = \{\theta \in \Theta | \operatorname{ord}_1\theta \le r_1, \dots, \operatorname{ord}_p\theta \le r_p\}$ and let $D_{r_1\dots r_p}$ denote the vector K-subspace of D generated by $\Theta(r_1, \dots, r_p)$. Setting $D_{r_1\dots r_p} = 0$ for $(r_1, \dots, r_p) \in \mathbf{Z}^p \setminus \mathbf{N}^p$, we obtain a family $\{D_{r_1\dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ called a *standard p-dimensional filtration* of D.

Definition 2.2. Let M be a left D-module. A family $\{M_{r_1\dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ of vector K-subspaces of M is called a p-dimensional filtration of M if

(i) for any fixed integers $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_p$ $(1 \le i \le p)$, $M_{r_1\dots r_i\dots r_p} \subseteq M_{r_1\dots r_{i-1}, r_i+1, r_{i+1}\dots r_p}$ and $M_{r_1\dots r_p} = 0$ for all sufficiently small $r_i \in \mathbf{Z}$;

(ii) $\bigcup\{M_{r_1\ldots r_p} | (r_1, \ldots, r_p) \in \mathbf{Z}^p\} = M$;
(iii) $D_{r_1\ldots r_p} M_{s_1\ldots s_p} \subseteq M_{r_1+s_1,\ldots,r_p+s_p}$ for any $(r_1, \ldots, r_p), (s_1, \ldots, s_p) \in \mathbf{Z}^p$.

If every vector K-space $M_{r_1\ldots r_p}$ is finitely generated and there exists an element $(h_1, \ldots, h_p) \in \mathbf{Z}^p$ such that $D_{r_1\ldots r_p} M_{h_1\ldots h_p} = M_{r_1+h_1,\ldots,r_p+h_p}$ for any $(r_1, \ldots, r_p) \in \mathbf{N}^p$, then the p-dimensional filtration is called *excellent*.

It is easy to see that if $u_1, \ldots, u_m$ is a finite system of generators of a left D-module M, then the filtration $\{\sum_{i=1}^{m} D_{r_1\ldots r_p} u_i \,|\, (r_1, \ldots, r_p) \in \mathbf{Z}^p\}$ is excellent.

Numerical polynomials

Definition 2.3. A polynomial $f(t_1, \ldots, t_p)$ in p variables $t_1, \ldots, t_p$ with rational coefficients is called *numerical* if $f(t_1, \ldots, t_p) \in \mathbf{Z}$ for all sufficiently large $t_1, \ldots, t_p \in \mathbf{Z}$, i.e., there exists an element $(s_1, \ldots, s_p) \in \mathbf{Z}^p$ such that $f(r_1, \ldots, r_p) \in \mathbf{Z}$ as soon as $(r_1, \ldots, r_p) \in \mathbf{Z}^p$ and $r_i \geq s_i$ for all $i = 1, \ldots, p$.

It is clear that every polynomial in several variables with integer coefficients is numerical. As an example of a numerical polynomial in p variables with non-integer coefficients one can consider the polynomial $\prod_{i=1}^{p} \binom{t_i}{n_i}$ $(n_1, \ldots, n_p \in \mathbf{Z})$, where $\binom{t}{k} = \frac{t(t-1)\ldots(t-k+1)}{k!}$ for any $k \in \mathbf{Z}, k \geq 1$, $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if $k < 0$.

If f is a numerical polynomials in p variables ($p > 1$), then $\deg f$ and $\deg_{t_i} f$ $(1 \leq i \leq p)$ will denote the total degree of f and the degree of f relative to the variable t_i, respectively. The following theorem proved in Kondrateva et al. (1992) gives the "canonical" representation of a numerical polynomial in several variables.

Theorem 2.4. *Let $f(t_1, \ldots, t_p)$ be a numerical polynomial in variables $t_1, \ldots, t_p$, and let $\deg_{t_i} f = n_i$ $(n_1, \ldots, n_p \in \mathbf{N})$. Then the polynomial $f(t_1, \ldots, t_p)$ can be represented in the form*

$$f(t_1, \ldots t_p) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} a_{i_1\ldots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p} \tag{2.2}$$

where all $a_{i_1\ldots i_p}$ are integers uniquely defined by the numerical polynomial.

In what follows, we deal with subsets of the set $\mathbf{N}^n$ where the positive integer n is represented as a sum of p non-negative integers $n_1, \ldots, n_p$ $(p \geq 1)$. In other words, we assume that a partition $(n_1, \ldots, n_p)$ of n is fixed.

If $A \subseteq \mathbf{N}^n$, then for any $r_1, \ldots, r_p \in \mathbf{N}$, $A(r_1, \ldots, r_p)$ will denote the subset of A that consists of all n-tuples $(a_1, \ldots, a_n)$ such that $a_1 + \cdots + a_{n_1} \leq r_1$, $a_{n_1+1} + \cdots + a_{n_1+n_2} \leq r_2, \ldots, a_{n_1+\cdots+n_{p-1}+1} + \cdots + a_n \leq r_p$. Furthermore, we associate with A a set $V_A = \{v = (v_1, \ldots, v_n) \in \mathbf{N}^n \,|\, v$ is not greater than or equal to any n-tuple in A with respect to the product order on $\mathbf{N}^n\}$. (Recall that the product order on $\mathbf{N}^n$ is a partial order $\leq_P$ such that $c = (c_1, \ldots, c_k) \leq_P c' = (c'_1, \ldots, c'_k)$ if and only if $c_i \leq c'_i$ for all $i = 1, \ldots, k$. If $c \leq_P c'$ and $c \neq c'$, we write $c <_P c'$.) Clearly, an n-tuple $v = (v_1, \ldots, v_n) \in \mathbf{N}^n$ lies in V_A if and only if for any element $(a_1, \ldots, a_n) \in A$ there exists $i \in \mathbf{N}$, $1 \leq i \leq n$ such that $a_i > v_i$.

The following two theorems proved in Kondrateva et al. (1992) generalize the well-known Kolchin's result on numerical polynomials of subsets of $\mathbf{N}^n$ (Kolchin, 1973, Ch. 0, Lemma 17) and give the explicit formula for the numerical polynomials in p variables associated with such a subset.

Theorem 2.5. *Let A be a subset of $\mathbf{N}^n$ where $n = n_1 + \cdots + n_p$ for some non-negative integers $n_1, \ldots, n_p$ $(p \geq 1)$. Then there exists a numerical polynomial $\omega_A(t_1, \ldots, t_p)$ with the following properties:*

(i) $\omega_A(r_1, \ldots, r_p) = \mathrm{Card}\, V_A(r_1, \ldots, r_p)$ *for all sufficiently large* $(r_1, \ldots, r_p) \in \mathbf{N}^p$ *(as usual,* Card M *denotes the number of elements of a finite set* M*).*
(ii) *The total degree of* ω_A *does not exceed* n *and* $\deg_{t_i} \omega_A \leq n_i$ *for* $i = 1, \ldots, p$.
(iii) $\deg \omega_A = n$ *if and only* $A = \emptyset$*. Then* $\omega_A(t_1, \ldots, t_p) = \prod_{i=1}^{p} \binom{t_i + n_i}{n_i}$.

Definition 2.6. The polynomial $\omega_A(t_1, \ldots, t_p)$ whose existence is stated by Theorem 2.5 is called the dimension polynomial of the set $A \subseteq \mathbf{N}^m$ associated with the partition $(n_1, \ldots, n_p)$ of n. If $p = 1$, the polynomial ω_A is called the Kolchin polynomial of the set A.

Theorem 2.7. *Let $A = \{a_1, \ldots, a_m\}$ $(m \geq 1)$ be a finite subset of $\mathbf{N}^n$ where $n = n_1 + \cdots + n_p$ for some non-negative integers $n_1, \ldots, n_p$ $(p \geq 1)$. Let $a_i = (a_{i1}, \ldots, a_{in})$ $(1 \leq i \leq m)$ and for any $l \in \mathbf{N}$, $0 \leq l \leq m$, let $\Gamma(l, m)$ denote the set of all l-element subsets of the set $\mathbf{N}_m = \{1, \ldots, m\}$. Furthermore, for any $\sigma \in \Gamma(l, m)$, let $\bar{a}_{\sigma j} = \max\{a_{ij} | i \in \sigma\}$ $(1 \leq j \leq n)$ and $b_{\sigma i} = \sum_{h=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} \bar{a}_{\sigma h}$ for $i = 1, \ldots, p$ (if $i = 1$, the sum is taken from 1 to n_1). Then*

$$\omega_A(t_1, \ldots, t_p) = \sum_{l=0}^{m} (-1)^l \sum_{\sigma \in \Gamma(l,m)} \prod_{i=1}^{p} \binom{t_i + n_i - b_{\sigma i}}{n_i}. \tag{2.3}$$

Remark 2.8. It is clear that if $A \subseteq \mathbf{N}^n$ and A' is the set of all minimal elements of A with respect to the product order, then the set A' is finite and $\omega_A(t_1, \ldots, t_p) = \omega_{A'}(t_1, \ldots, t_p)$. Thus, Theorem 2.7 gives an algorithm that allows one to find a numerical polynomial associated with any subset of $\mathbf{N}^n$ (and with a given partition of n): one should first find the set of all minimal points of the subset and then apply Theorem 2.7.

Differential field extensions

Recall that a *differential ring* is a commutative ring R considered together with a finite set Δ of mutually commuting derivations of the ring R into itself. The set Δ is called a basic set of the differential ring R which is also called a Δ-ring. A subring (ideal) R_0 of a Δ-ring R is called a differential or a Δ-subring of R (respectively, a differential or a Δ-ideal of R) if R_0 is closed with respect to the action of any operator $\delta \in \Delta$. If a differential (Δ-)ring is a field, it is called a differential (or Δ-)field.

A subfield K_0 of Δ-field K is said to be a differential (or Δ-) subfield of K if $\delta(K_0) \subseteq K_0$ for any $\delta \in \Delta$. If K_0 is a Δ-subfield of Δ-field K and $\Sigma \subseteq K$, then the intersection of all Δ-subfields of K containing K_0 and Σ is the unique Δ-subfield of K containing K_0 and Σ and contained in every Δ-subfield of K containing K_0 and Σ. It is denoted by $K_0\langle\Sigma\rangle$. If $K = K_0\langle\Sigma\rangle$ and the set Σ is finite, $\Sigma = \{\eta_1, \ldots, \eta_n\}$, then K is said to be a finitely generated Δ-extension of K_0 with the set of Δ-generators $\{\eta_1, \ldots, \eta_n\}$. In this case we write $K = K_0\langle\eta_1, \ldots, \eta_n\rangle$. It is easy to see that $K_0\langle\eta_1, \ldots, \eta_n\rangle$ coincides with the field $K_0(\{\theta\eta_i | \theta \in \Theta_\Delta, 1 \leq i \leq n\})$ where Θ_Δ is the free commutative semigroup generated by the set Δ.

Now we can formulate the fundamental Kolchin's theorem on differential dimension polynomial (Kolchin, 1964).

Theorem 2.9. *Let K be a differential field of zero characteristic with a basic set of derivation operators $\Delta = \{\delta_1, \ldots, \delta_m\}$. Let Θ_Δ be the free commutative semigroup generated by Δ,* *and for any* $r \in \mathbf{N}$, *let* $\Theta_{\Delta}(r) = \{\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta_{\Delta} \mid \sum_{i=1}^{m} k_i \le r\}$. *Furthermore, let* L *be a differential field extension of* K *generated by a finite set* $\eta = \{\eta_1, \dots, \eta_n\}$. *Then there exists a polynomial* $\omega_{\eta|K}(t)$ *in one variable* t *with rational coefficients (called a* **differential dimension polynomial** *of the extension) such that*

(i) $\omega_{\eta|K}(r) = \operatorname{trdeg}_K K(\{\theta\eta_j \mid \theta \in \Theta_{\Delta}(r), 1 \le j \le n\})$ *for all sufficiently large integers* r;

(ii) $\deg \omega_{\eta|K} \le m$ *and the polynomial* $\omega_{\eta|K}(t)$ *can be written as* $\omega_{\eta|K}(t) = \sum_{i=1}^{m} a_i \binom{t+i}{i}$ *where* $a_0, \dots, a_m$ *are some integers;*

(iii) $d = \deg \omega_{\eta|K}$ *and the coefficients* a_m *and* a_d *are differential birational invariants of the extension* L/K, *that is, they do not depend on the choice of the system of* Δ*-generators* η *of* L/K. *(Clearly,* $a_d \ne a_m$ *if and only if* $d < m$, *that is* $a_m = 0$.*) Moreover,* a_m *is equal to the differential transcendence degree of* L *over* K, *i.e., to the maximal number of elements* $\xi_1, \dots, \xi_k \in L$ *such that the set* $\{\theta\xi_i \mid \theta \in \Theta_{\Delta}, 1 \le i \le k\}$ *is algebraically independent over* K.

3. Reduction. Gröbner bases with respect to several term orderings

With the notation introduced at the beginning of the preceding section, let E be a finitely generated free module over the Ore polynomial ring $D = K[X]$ ($X = \{x_1, \dots, x_n\} = X_1 \bigcup \dots \bigcup X_p$, where $X_1 = \{x_1, \dots, x_{n_1}\}, X_2 = \{x_{n_1+1}, \dots, x_{n_1+n_2}\}, \dots, X_p = \{x_{n_1+\dots+n_{p-1}+1}, \dots, x_n\}$ and $n_1 + \dots + n_p = n$). Let $e_1, \dots, e_m$ be a fixed basis of E over D. Then elements of E of the form θe_k ($\theta \in \Theta, 1 \le k \le m$) will be called *terms* and the set of all terms of E will be denoted by Θe. The order of a term θe_k and the order of this term with respect to X_i ($1 \le i \le p$) are defined as the order of the monomial θ and the order of θ relative to X_i, respectively. A term θe_i is said to be a *multiple* of a term $\theta' e_j$ if $i = j$ and θ' divides θ in the semigroup Θ. Then we also say that the term $\theta' e_j$ divides θe_i and write $\theta' e_j | \theta e_i$. The least common multiple of two terms $u = \theta_1 e_i$ and $v = \theta_2 e_j$ is defined as usual: $lcm(u, v) = lcm(\theta_1, \theta_2)e_i$ if $i = j$ and $lcm(u, v) = 0$ if $i \ne j$.

We shall consider p orderings of the set Θe that correspond to the orderings of the set Θ introduced in Section 2. These ordering are denoted by the same symbols $<_1, \dots, <_p$ and defined as follows: if $\theta e_k, \theta' e_l \in \Theta e$, then $\theta e_k <_i \theta' e_l$ if and only if $\theta <_i \theta'$ in Θ or $\theta = \theta'$ and $k < l$.

Since the set Θe is a basis of the vector K-space E, every nonzero element $f \in E$ has a unique representation in the form

$$f = a_1\theta_1 e_{i_1} + \dots + a_l\theta_l e_{i_l} \tag{3.1}$$

where $\theta_i \in \Theta$, $a_i \in K$, $a_i \ne 0$ ($1 \le i \le n$), and the terms $\theta_1 e_{i_1}, \dots, \theta_l e_{i_l}$ are distinct ($1 \le i_1, \dots, i_l \le m$).

Definition 3.1. Let f be a nonzero element of the module E written in the form (3.1) and let $\theta_k e_{i_k}$ ($1 \le k \le p$) be the greatest term of the set $\{\theta_1 e_{i_1}, \dots, \theta_l e_{i_l}\}$ with respect to the order $<_j$ ($1 \le j \le p$). Then the term $\theta_k e_{i_k}$ is called the j-leader of the element f; it is denoted by $u_f^{(j)}$. (Of course, it is possible that $u_f^{(i)} = u_f^{(j)}$ for some distinct numbers i and j.) The non-negative integer $\operatorname{ord}_j u_f^{(j)}$ is called the jth order of f and denoted by $\operatorname{ord}_j f$ ($j = 1, \dots, p$). The coefficient of $u_f^{(j)}$ in f is said to be the j-leading coefficient of f; it is denoted by $lc_j(f)$.

Definition 3.2. Let $f, g \in E$, where g is a nonzero element, and let $k, i_1, \dots, i_r$ be distinct elements of the set $\{1, \dots, p\}$. Then f is said to be $(<_k, <_{i_1}, \dots, <_{i_r})$*-reduced* with respect to g if f does not contain any multiple $\theta u_g^{(k)}$ $(\theta \in \Theta)$ such that $\mathrm{ord}_{i_\nu}(\theta u_g^{(i_\nu)}) \le \mathrm{ord}_{i_\nu} u_f^{(i_\nu)}$ $(\nu = 1, \dots, r)$. An element $f \in E$ is said to be $(<_k, <_{i_1}, \cdots <_{i_r})$-reduced with respect to a set $G \subseteq E$, if f is $(<_k, <_{i_1}, \cdots <_{i_r})$-reduced with respect to every element of G.

Let us consider $p-1$ new symbols $z_1, \dots, z_{p-1}$ and the free commutative semigroup Γ of all power products $\gamma = x_1^{k_1} \dots x_n^{k_n} z_1^{l_1} \dots z_{p-1}^{l_{p-1}}$ with non-negative integer exponents. Let Γe denote the set of terms $\{\gamma e_j | \gamma \in \Gamma, 1 \le j \le m\} = \Gamma \times \{e_1, \dots, e_m\}$ (with the division as in the set Θe above). For any nonzero element $f \in E$, let $d_i(f) = \mathrm{ord}_i u_f^{(i)} - \mathrm{ord}_i u_f^{(1)}$ $(2 \le i \le p)$ and let $\rho : E \to \Gamma e$ be defined by $\rho(f) = z_1^{d_2(f)} \dots z_{p-1}^{d_{p(f)}} u_f^{(1)}$.

Definition 3.3. With the above notation, let N be a D-submodule of E. A finite set of nonzero elements $G = \{g_1, \dots, g_r\} \subseteq N$ is called a Gröbner basis of N with respect to the orders $<_1, \dots, <_p$ if for any nonzero element $f \in N$, there exists $g_i \in G$ such that $\rho(g_i) | \rho(f)$.

It is clear that every Gröbner basis of N with respect to the orders $<_1, \dots, <_p$ is a Gröbner basis of N with respect to $<_1$ in the usual sense. Therefore, every Gröbner basis of N with respect to the orders $<_1, \dots, <_p$ generates N as a left D-module. A set $\{g_1, \dots, g_r\} \subseteq E$ is said to be a *Gröbner basis with respect to the orders* $<_1, \dots, <_p$ if G is a Gröbner basis of $N = \sum_{i=1}^r Dg_i$ with respect to $<_1, \dots, <_p$.

Definition 3.4. Given $f, g, h \in E$, with $g \neq 0$, we say that the element f $(<_k, <_{i_1}, \cdots <_{i_l})$-*reduces* to h *modulo* g in one step and write $f \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g} h$ if and only if f contains some term w with a coefficient a such that $u_g^{(k)} | w$,

$$h = f - a \left(\tau_{\frac{w}{u_g^{(k)}}} (lc_k(g)) \right)^{-1} \frac{w}{u_g^{(k)}} g$$

and $\mathrm{ord}_{i_\nu} \frac{w}{u_g^{(k)}} u_g^{(i_\nu)} \le \mathrm{ord}_{i_\nu} u_f^{(i_\nu)}$ for $\nu = 1, \dots, l$.

Definition 3.5. Let $f, h \in E$ and let $G = \{g_1, \dots, g_r\}$ be a finite set of non-zero elements of E. We say that f $(<_k, <_{i_1}, \dots, <_{i_l})$-*reduces* to h *modulo* G and write $f \xrightarrow[<_k; <_{i_1}, \dots, <_{i_l}]{G} h$ if and only if there exists a sequence of elements $g^{(1)}, g^{(2)}, \dots, g^{(q)} \in G$ and a sequence of elements $h_1, \dots, h_{q-1} \in E$ such that $f \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(1)}} h_1 \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(2)}} \dots \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(q-1)}}$ $h_{q-1} \xrightarrow[<_k, <_{i_1}, \dots <_{i_l}]{g^{(q)}} h$.

Theorem 3.6. *With the above notation, let f be an element of E and let $G = \{g_1, \dots, g_r\} \subseteq E$ be a Gröbner basis with respect to the orders $<_1, \dots, <_p$ on Θe. Then there exist elements $g \in E$ and $Q_1, \dots, Q_r \in D$ such that $f - g = \sum_{i=1}^r Q_i g_i$ and g is $(<_1, \dots, <_p)$-reduced with respect to G.*

Proof. If f is $(<_1, \dots, <_p)$-reduced with respect to G, the statement is obvious (one can set $g = f$). Suppose that f is not reduced with respect to G. Let $u_i^{(j)} = u_{g_i}^{(j)}$ $(1 \le i \le r, 1 \le j \le p)$

and let a_i be the coefficient of the term $u_i^{(1)}$ in g_i ($i = 1, \dots, r$). In what follows, a term w_h, that appears in an element $h \in E$, will be called a G-leader of h if w_h is the greatest (with respect to the order $<_1$) term among all terms $\theta u_i^{(1)}$ ($\theta \in \Theta, 1 \le i \le r$) that appear in h and satisfy the condition $\mathrm{ord}_j(\theta u_i^{(j)}) \le \mathrm{ord}_j u_h^{(j)}$ for $j = 2, \dots, p$.

Let w_f be the G-leader of the element f and let c_f be the coefficient of w_f in f. Then $w_f = \theta u_i^{(1)}$ for some $\theta \in \Theta$, $1 \le i \le r$, such that $\mathrm{ord}_j(\theta u_i^{(j)}) \le \mathrm{ord}_j u_f^{(j)}$ for $j = 2, \dots, p$. Without loss of generality we may assume that i corresponds to the maximum (with respect to the order $<_1$) 1-leader $u_i^{(1)}$ in the set of all such 1-leaders of elements of G. Let us consider the element $f' = f - c_f(\tau_\theta(a_i))^{-1}\theta g_i$. Obviously, f' does not contain w_f and $\mathrm{ord}_j(u_{f'}^{(j)}) \le \mathrm{ord}_j u_f^{(j)}$ for $j = 2, \dots, p$. Furthermore, f' cannot contain any term $\theta' u_i^{(1)}$ ($\theta' \in \Theta$, $1 \le i \le r$) which is greater than w_f with respect to $<_1$ and satisfies the condition $\mathrm{ord}_j(\theta' u_i^{(j)}) \le \mathrm{ord}_j u_{f'}^{(j)}$ for $j = 2, \dots, p$. Indeed, if the last inequality holds, then $\mathrm{ord}_j(\theta' u_i^{(j)}) \le \mathrm{ord}_j u_f^{(j)}$ ($2 \le j \le p$) , so that $\theta' u_i^{(1)}$ cannot appear in f. This term cannot appear in θg_i either, since $u_{\theta g_i}^{(1)} = \theta u_{g_i}^{(1)} = w_f <_j \theta' u_i^{(1)}$. Thus, $\theta' u_i^{(1)}$ cannot appear in f', whence the G-leader of f' is strictly less (with respect to $<_1$) than the G-leader of f. Applying the same procedure to the element f' and continuing in the same way, we obtain an element $g \in E$ such that $f - g$ is a linear combination of elements $g_1, \dots, g_r$ with coefficients in D and g is reduced with respect to G. This completes the proof. □

The process of reduction described in the proof of the last theorem can be realized by the following algorithm. (It can be used for the reduction with respect to any finite set of elements of the free D-module E.)

Algorithm 1. $(f, r, g_1, \dots, g_r; g; Q_1, \dots, Q_r)$

Input: $f \in E$, a positive integer r, $G = \{g_1, \dots, g_r\} \subseteq E$ where $g_i \ne 0$ for $i = 1, \dots, r$
Output: Elements $g \in E$ and $Q_1, \dots, Q_r \in D$ such that $g = f - \sum_{i=1}^r Q_i g_i$ and g is reduced with respect to G
Begin $Q_1 := 0, \dots, Q_r := 0, g := f$
While there exist i, $1 \le i \le r$, and a term w, that appears in g_i with a nonzero coefficient $c(w)$, such that $u_{g_i}^{(1)} | w$ and $\mathrm{ord}_j(\frac{w}{u_{g_i}^{(1)}} u_{g_i}^{(j)}) \le \mathrm{ord}_j u_g^{(j)}$ for $j = 2, \dots, p$
do
z:= the greatest (with respect to $<_1$) of the terms w that satisfy the above conditions.
k:= the smallest number i for which $u_{g_i}^{(1)}$ is the greatest (with respect to $<_1$) 1-leader of an element $g_i \in G$ such that $u_{g_i}^{(1)} | z$ and $\mathrm{ord}_j(\frac{z}{u_{g_i}^{(1)}} u_{g_i}) \le \mathrm{ord}_j u_g^{(j)}$ for $j = 2, \dots, p$.

$$Q_k := Q_k + c(z)\left(\tau_{\frac{z}{u_{gk}^{(1)}}}(lc_1(g_k))\right)^{-1} \frac{z}{u_{gk}^{(1)}};\ g := g - c(z)\left(\tau_{\frac{z}{u_{gk}^{(1)}}}(lc_1(g_k))\right)^{-1} \frac{z}{u_{gk}^{(1)}} g_k.$$

End

The proof of Theorem 3.6 shows that if G is a Gröbner basis of a D-submodule N of E, then a reduction step described in Definition 3.4 (with some $g \in G$) can be applied to every nonzero element $f \in N$. As a result of such a step, we obtain an element of N whose G-leader is strictly less (with respect to $<_1$) than the G-leader of f. This observation leads to the following statement.

Proposition 3.7. *Let $G = \{g_1, \dots, g_r\}$ be a Gröbner basis of a D-submodule N of E with respect to the orders $<_1, \dots, <_p$. Then*

(i) *$f \in N$ if and only if $f \xrightarrow[<_1,<_2,\cdots<_p]{G} 0$.*

(ii) *If $f \in N$ and f is $(<_1, <_2, \cdots <_p)$-reduced with respect to G, then $f = 0$.* $\square$

Definition 3.8. Let f and g be two elements in the free D-module E, $k \in \{1, \dots, p\}$ and $\theta_f = \frac{lcm(u_f^{(k)}, u_g^{(k)})}{u_f^{(k)}}$, $\theta_g = \frac{lcm(u_f^{(k)}, u_g^{(k)})}{u_g^{(k)}}$. Then the element $S_k(f, g) = \left(\tau_{\theta_f}(lc_k(f))\right)^{-1} \theta_f f - \left(\tau_{\theta_g}(lc_k(g))\right)^{-1} \theta_g g$ is called the kth *S-polynomial* of f and g.

The proof of the following proposition is similar to the proof of the corresponding statement for a polynomial ring over a field, see Adams and Loustaunau (1994, Lemma 1.7.5).

Proposition 3.9. *$0 \neq f, g_1, \dots, g_r \in E$ ($r \geq 1$) and let $f = \sum_{i=1}^{r} c_i \omega_i g_i$ where $\omega_i \in \Theta$, $c_i \in K$ ($1 \leq i \leq r$). Let $k \in \{1, \dots, p\}$ and for any $\nu, j \in \{1, \dots, r\}$, let $u_{\nu j}^{(k)} = lcm(u_{g_\nu}^{(k)}, u_{g_j}^{(k)})$. Furthermore, suppose that $\omega_1 u_{g_1}^{(k)} = \cdots = \omega_r u_{g_r}^{(k)} = u$ for some $k \in \{1, \dots, p\}$, $u_f^{(k)} <_k u$ and there is a non-empty set $I \subseteq \{1, \dots, p\} \setminus \{k\}$ such that $\omega_i u_{g_i}^{(l)} \leq_l u_f^{(l)}$ for all $i \in \{1, \dots, r\}, l \in I$. Then there exist elements $c_{\nu j} \in K$ ($1 \leq \nu \leq s, 1 \leq j \leq t$) such that $f = \sum_{\nu=1}^{s} \sum_{j=1}^{t} c_{\nu j} \theta_{\nu j} S_k(g_\nu, g_j)$ where $\theta_{\nu j} = \frac{u}{u_{\nu j}^{(k)}}$ and $\theta_{\nu j} u_{S_k(g_\nu, g_j)}^{(k)} <_k u$, $\theta_{\nu j} u_{S_k(g_\nu, g_j)}^{(l)} \leq_l u_f^{(l)}$ ($1 \leq \nu \leq s, 1 \leq j \leq t, l \in I$).* $\square$

The following result provides the theoretical foundation for the algorithm of constructing Gröbner bases with respect to several term orders.

Theorem 3.10. *With the above notation, let $G = \{g_1, \dots, g_r\}$ be a Gröbner basis of a D-submodule N of E with respect to each of the following sequences of orders: $<_p$; $<_{p-1}, <_p$; $\dots$; $<_{k+1}, \dots, <_p$ ($1 \leq k \leq p - 1$). Furthermore, suppose that*

$$S_k(g_i, g_j) \xrightarrow[<_k, <_{k+1}, \dots, <_p]{G} 0 \quad \textit{for any } g_i, g_j \in G.$$

Then G is a Gröbner basis of N with respect to $<_k, <_{k+1}, \dots, <_p$.

Proof. Notice that it is sufficient to prove that under the conditions of the theorem every element $f \in N$ can be represented as

$$f = \sum_{i=1}^{r} h_i g_i \tag{3.2}$$

where $h_1, \dots, h_r \in D$,

$$\max_{<_k} \{u_{h_i}^{(k)} u_{g_i}^{(k)} \mid 1 \leq i \leq r\} = u_f^{(k)} \tag{3.3}$$

and

$$\operatorname{ord}_j (u_{h_i}^{(j)} u_{g_i}^{(j)}) \leq \operatorname{ord}_j u_f^{(j)} \quad (j = k + 1, \dots, p). \tag{3.4}$$

(The symbol $\max_{<_k}$ in (3.3) means the maximum with respect to $<_k$.) Indeed, with the notation of Definition 3.3, in this case $\rho(f)$ is divisible by $\rho(g_i)$ where g_i gives the maximum value in the left-hand side of (3.3).

We proceed by induction on $p - k$. If $p - k = 0$, our statement is a classical result of the theory of Gröbner bases (the fact that we consider modules over D rather than modules over a usual polynomial ring is not essential). Let $k < p$ and let $f \in N$. By the induction hypothesis, f can be written as

$$f = \sum_{i=1}^{r} H_i g_i \tag{3.5}$$

where $H_1, \dots, H_r \in D$,

$$\max\nolimits_{<_{k+1}}\{u_{H_i}^{(k+1)} u_{g_i}^{(k+1)} \mid 1 \le i \le r\} = u_f^{(k+1)} \tag{3.6}$$

and

$$\operatorname{ord}_j(u_{H_i}^{(j)} u_{g_i}^{(j)}) \le \operatorname{ord}_j u_f^{(j)} \quad (j = k+2, \dots, p;\ i = 1, \dots, r). \tag{3.7}$$

Let us choose among all representations of f in the form (3.5) with conditions (3.6) and (3.7) a representation with the smallest (with respect to $<_k$) possible term $u = \max_{<_k}\{u_{H_i}^{(k)} u_{g_i}^{(k)} \mid 1 \le i \le r\}$. Setting $d_i = lc_k(H_i)\ (1 \le i \le r)$ and breaking the sum in (3.5) in two parts, we can write

$$f = \sum_{u_{H_i}^{(k)} u_{g_i}^{(k)} = u} d_i u_{H_i}^{(k)} g_i + \sum_{u_{H_i}^{(k)} u_{g_i}^{(k)} = u} (H_i - d_i u_{H_i}^{(k)}) g_i + \sum_{u_{H_i}^{(k)} u_{g_i}^{(k)} <_k u} H_i g_i. \tag{3.8}$$

Notice that if $u = u_f^{(k)}$, then the expansion (3.8) satisfies conditions (3.2)–(3.4). Indeed, by (3.6) we have $u_f^{(k+1)} = \max_{<_{k+1}}\{u_{H_i}^{(k+1)} u_{g_i}^{(k+1)} \mid 1 \le i \le r\}$, hence $\max\{\operatorname{ord}_{k+1}(u_{H_i}^{(k+1)} u_{g_i}^{(k+1)}) \mid 1 \le i \le r\} \le \operatorname{ord}_{k+1} u_f^{(k+1)}$ and $\operatorname{ord}_j(u_{H_i}^{(j)} u_{g_i}^{(j)}) \le \operatorname{ord}_j u_f^{(j)}$ for $j = k+2, \dots, p;\ i = 1, \dots, r$ (see (3.7)).

Suppose that $u_f^{(k)} <_k u$. Since $u = \max_{<_k}\{u_{H_i}^{(k)} u_{g_i}^{(k)} \mid 1 \le i \le r\}$, we have $u_{H_i - d_i u_{H_i}(k)}^{(k)} <_k u$ $(1 \le i \le r)$, whence the k-leader of the first sum in (3.8) does not exceed u with respect to $<_k$. Furthermore, it is clear that $u_{H_i}^{(k)} u_{g_i}^{(k)} = u$ for any term in the sum

$$\tilde{f} = \sum_{u_{H_i}^{(k)} u_{g_i}^{(k)} = u} d_i u_{H_i}^{(k)} g_i \tag{3.9}$$

and $\operatorname{ord}_j u_{\tilde{f}}^{(j)} \le \max_{i \in I}\{\operatorname{ord}_j(u_{H_i}^{(k)} u_{g_i}^{(j)})\} \le \max_{i \in I}\{\operatorname{ord}_j(u_{H_i}^{(j)} u_{g_i}^{(j)})\} \le \operatorname{ord}_j u_f^{(j)}$ $(j = k+1, \dots, p$ and I denotes the set of all indices $i \in \{1, \dots, r\}$ that appear in (3.9)).

Let $u_{\nu j}^{(k)} = lcm(u_{g_\nu}^{(k)}, u_{g_j}^{(k)})$ for any $\nu, j \in I$, $\nu \ne j$ and let $\theta_{\nu j} = \frac{u}{u_{\nu j}^{(k)}} \in \Theta$. (Since $u = u_{H_i}^{(k)} u_{g_i}^{(k)}$ for every $i \in I$, $u_{\nu j}^{(k)} | u$.) By Proposition 3.9, there exist elements $c_{\nu j} \in K$ such that

$$\tilde{f} = \sum_{\nu, j} c_{\nu j} \theta_{\nu j} S_k(g_\nu, g_j) \tag{3.10}$$

where $u_{\theta_{\nu j} S_k(g_\nu, g_j)}^{(k)} <_k u_{\tilde{f}}^{(k)} = u$ and $\operatorname{ord}_j u_{\theta_{\nu j} S_k(g_\nu, g_j)}^{(j)} \le \operatorname{ord}_j u_{\tilde{f}}^{(j)}$ $(j = k+1, \dots, p)$. Since $S_k(g_\nu, g_j) \xrightarrow[<_k, <_{k+1}, \cdots <_p]{G} 0$, there exist $q_{i\nu j} \in D$ such that $S_k(g_\nu, g_j) = \sum_{i=1}^{r} q_{i\nu j} g_i$ and $u_{q_{i\nu j}}^{(k)} u_{g_i}^{(k)} \le_k u_{S_k(g_\nu, g_j)}^{(k)}$, $\operatorname{ord}_l(u_{q_{i\nu j}}^{(l)} u_{g_i}^{(l)}) \le_l \operatorname{ord}_l u_{S_k(g_\nu, g_j)}^{(l)}$ for $l = k+1, \dots, p$.

Thus, for any indices ν, j in the sum (3.10) we have $\theta_{\nu j} S_k(g_\nu, g_j) = \sum_{i=1}^{r} (\theta_{\nu j} q_{i\nu j}) g_i$ where $u^{(k)}_{\theta_{\nu j} q_{i\nu j}} u^{(k)}_{g_i} = \theta_{\nu j} u^{(k)}_{q_{i\nu j}} u^{(k)}_{g_i} \leq_k \theta_{\nu j} u^{(k)}_{S_k(g_\nu, g_j)} <_k u$. It follows that

$$\tilde{f} = \sum_{\nu,j} c_{\nu j} \sum_{i=1}^{r} (\theta_{\nu j} q_{i\nu j}) g_i = \sum_{i=1}^{r} \left(\sum_{\nu,j} c_{\nu j} \theta_{\nu j} q_{i\nu j} \right) g_i = \sum_{i=1}^{r} \tilde{H}_i g_i \tag{3.11}$$

where $\tilde{H}_i = \sum_{\nu,j} c_{\nu j} \theta_{\nu j} q_{i\nu j}$ $(1 \leq i \leq r)$ and $u^{(k)}_{\tilde{H}_i} u^{(k)}_{g_i} <_k u$ $(1 \leq i \leq r)$.

Furthermore, for any $l = k+1, \ldots, p$, we have $\mathrm{ord}_l(u^{(l)}_{\tilde{H}_i} u^{(l)}_{g_i}) \leq \max_{\nu,j}\{\mathrm{ord}_l(\theta_{\nu j} u^{(l)}_{\tilde{H}_i} u^{(l)}_{g_i})\} \leq \max_{\nu,j}\{\max\{\mathrm{ord}_l(\theta_{\nu j} \frac{u^{(k)}_{\nu j}}{u^{(k)}_{g_\nu}} u^{(l)}_{g_\nu}), \mathrm{ord}_l(\theta_{\nu j} \frac{u^{(k)}_{\nu j}}{u^{(k)}_{g_j}} u^{(l)}_{g_j})\}\} = \max_{\nu,j}\{\max\{\mathrm{ord}_l(\frac{u}{u^{(k)}_{g_\nu}} u^{(l)}_{g_\nu}), \mathrm{ord}_l(\frac{u}{u^{(k)}_{g_j}} u^{(l)}_{g_j})\}\}$ $\leq \mathrm{ord}_l u = \mathrm{ord}_l u^{(k)}_{\tilde{f}} \leq \mathrm{ord}_l u^{(l)}_{\tilde{f}}$, so that representation (3.11) satisfies the condition

$$\mathrm{ord}_l(u^{(l)}_{\tilde{H}_i} u^{(l)}_{g_i}) \leq \mathrm{ord}_l u^{(l)}_{\tilde{f}} \tag{3.12}$$

for $i = 1, \ldots, r$. Substituting (3.11) into (3.8) we obtain

$$f = \sum_{i=1}^{r} \tilde{H}_i g_i + \sum_{u^{(k)}_{H_i} u^{(k)}_{g_i} = u} (H_i - d_i u^{(k)}_{H_i}) g_i + \sum_{u^{(k)}_{H_i} u^{(k)}_{g_i} <_k u} H_i g_i \tag{3.13}$$

where, denoting each $H_i - d_i u^{(k)}_{H_i}$ in the second sum by H'_i, we have the following conditions: $u^{(k)}_{\tilde{H}_i} u^{(k)}_{g_i} <_k u$, $u^{(k)}_{H'_i} u^{(k)}_{g_i} <_k u$ for any term with index i in the second sum in (3.13), and $u^{(k)}_{H_i} u^{(k)}_{g_i} <_k u$ for any term with index i in the third sum in (3.13). Also, for every $l = k+1, \ldots, p$, the inequality (3.12) implies $\mathrm{ord}_l(u^{(l)}_{\tilde{H}_i} u^{(l)}_{g_i}) \leq \mathrm{ord}_l u^{(l)}_{\tilde{f}}$. Therefore,

$$\mathrm{ord}_l(u^{(l)}_{\tilde{H}_i} u^{(l)}_{g_i}) \leq \max\{\mathrm{ord}_l(u^{(k)}_{H_i} u^{(l)}_{g_i})\} \leq \max\{\mathrm{ord}_l(u^{(l)}_{H_i} u^{(l)}_{g_i})\} \leq \mathrm{ord}_l u^{(l)}_{f}$$

where the maxima are taken over the set of all indices i in the first sum in (3.8). Furthermore, (3.7) implies that for $l = k+1, \ldots, p$ we have $\mathrm{ord}_l(u^{(l)}_{H'_i} u^{(l)}_{g_i}) \leq \mathrm{ord}_l(u^{(k)}_{H_i} u^{(l)}_{g_i}) \leq \mathrm{ord}_l u^{(l)}_{f}$ for every index i in the second sum in (3.13) and $\mathrm{ord}_l(u^{(l)}_{H_i} u^{(l)}_{g_i}) \leq \mathrm{ord}_l u^{(l)}_{f}$ for every index i in the third sum in (3.13). Thus, (3.13) is a representation of f in the form (3.5) with conditions (3.6) and (3.7) such that if one writes (3.13) as $f = \sum_{i=1}^{r} \tilde{H}'_i f_i$ (combining the sums in (3.13)), then $\max\{u^{(k)}_{\tilde{H}'_1} u^{(k)}_{g_1}, \ldots, u^{(k)}_{\tilde{H}'_r} u^{(k)}_{g_r}\} <_k u$ and one has conditions of the types (3.6) and (3.7). We have arrived at a contradiction with our choice of representation (3.5) with conditions (3.6), (3.7) and the smallest (with respect to $<_k$) possible value of $\max\{u^{(k)}_{\tilde{H}'_i} u^{(k)}_{g_i} \mid 1 \leq i \leq r\} = u$. Thus, every element $f \in N$ can be written in the form (3.2) with conditions (3.3) and (3.4). $\square$

The last theorem allows one to construct a Gröbner basis of a D-module $N \subseteq E$ with respect to $<_1, \ldots, <_p$ starting with a Gröbner basis of N with respect to $<_p$. The corresponding algorithm is just a p-step procedure where each step is a version of the classical Buchberger algorithm. (The first step is the Buchberger algorithm with respect to $<_p$, and for $k = 1, \ldots, p-1$, the kth step is an analog of the Buchberger algorithm for $(p-k)$th S-polynomials and reduction $\xrightarrow[<_k, <_{k+1}, \cdots <_p]{G}$.)

4. Multivariable dimension polynomials of D-modules

In this section we use properties of Gröbner bases with respect to several term orderings to prove the existence and determine invariants of multivariable dimension polynomials of finitely generated D-modules. The following result can be considered as the main step in this direction.

Theorem 4.1. *Let D be the ring of Ore polynomials over a field K, M a D-module generated by a finite set $\{f_1, \dots, f_m\}$, and E a free left D-module with free generators $e_1, \dots, e_m$. Let $\pi : E \longrightarrow M$ be the natural D-epimorphism ($\pi(e_i) = f_i$ for $i = 1, \dots, m$), $N = \mathrm{Ker}\,\pi$, and $G = \{g_1, \dots, g_d\}$ a Gröbner basis of N with respect to $<_1, \dots, <_p$. Furthermore, for any $(r_1, \dots, r_p) \in \mathbf{N}^p$, let $M_{r_1 \dots r_p} = \sum_{i=1}^{m} D_{r_1 \dots r_p} f_i$, $V_{r_1 \dots r_p} = \{u \in \Theta e \mid \mathrm{ord}_i u \le r_i\ (1 \le i \le p)$ and $u \ne \theta u_g^{(1)}$ for any $\theta \in \Theta, g \in G\}$, $W_{r_1 \dots r_p} = \{u \in \Theta e \setminus V_{r_1 \dots r_p} | \mathrm{ord}_i u \le r_i$ for $i = 1, \dots, p$ and for every $\theta \in \Theta, g \in G$ such that $u = \theta u_g^{(1)}$, there exists $i \in \{2, \dots, p\}$ such that $\mathrm{ord}_i \theta u_g^{(i)} > r_i\}$, and $U_{r_1 \dots r_p} = V_{r_1 \dots r_p} \bigcup W_{r_1 \dots r_p}$. Then for any $(r_1, \dots, r_p) \in \mathbf{N}^p$, the set $\pi(U_{r_1 \dots r_p})$ is a basis of $M_{r_1 \dots r_p}$ over K.*

Proof. Let us prove, first, that if an element θf_i $(1 \le i \le m, \theta \in \Theta(r_1, \dots, r_p))$ does not belong to $\pi(U_{r_1 \dots r_p})$, then it is a linear combination of elements of $\pi(U_{r_1 \dots r_p})$ with coefficients in K (so that the set $\pi(U_{r_1 \dots r_p})$ generates the vector K-space $M_{r_1 \dots r_p}$). Indeed, since $\theta f_i \notin \pi(U_{r_1 \dots r_p})$, $\theta e_i \notin U_{r_1 \dots r_p}$ whence $\theta e_i = \theta' u_{g_j}^{(1)}$ for some $\theta' \in \Theta$, $1 \le j \le d$, such that $\mathrm{ord}_\nu(\theta' u_{g_j}^{(\nu)}) \le r_\nu$ $(\nu = 2, \dots, p)$. Let us consider the element $g_j = a_j u_{g_j}^{(1)} + \cdots$ $(a_j \in K, a_j \ne 0)$, where dots are placed instead of the sum of the other terms of g_j with non-zero coefficients (obviously, those terms are less than $u_{g_j}^{(1)}$ with respect to the order $<_1$). Since $g_j \in N = \mathrm{Ker}\,\pi$, $\pi(g_j) = a_j \pi(u_{g_j}^{(1)}) + \cdots = 0$, whence $\pi(\theta' g_j) = a_j \pi(\theta' u_{g_j}^{(1)}) + \cdots = a_j \pi(\theta e_i) + \cdots = a_j \theta f_i + \cdots = 0$, so that θf_i is a finite linear combination with coefficients in K of some elements $\tilde{\theta}_l f_l$ $(1 \le l \le m)$ such that $\tilde{\theta}_l \in \Theta(r_1, \dots, r_p)$ and $\tilde{\theta}_l e_l <_1 \theta' u_{g_j}^{(1)}$. ($\mathrm{ord}_1 \tilde{\theta}_l \le r_1$, since $\tilde{\theta}_l e_l <_1 \theta e_i$ and $\theta \in \Theta(r_1, \dots, r_p)$; $\mathrm{ord}_\nu \tilde{\theta}_l \le r_\nu$ $(\nu = 2, \dots, p)$, because $\tilde{\theta}_l e_l \le_\nu u_{\theta' g_j}^{(\nu)} = \theta' u_{g_j}^{(\nu)}$ and $\mathrm{ord}_\nu(\theta' u_{g_j}^{(\nu)}) \le r_\nu$.) Thus, we can apply the induction on θe_j $(\theta \in \Theta, 1 \le j \le m)$ with respect to the order $<_1$ and obtain that every element θf_i $(\theta \in \Theta(r_1, \dots, r_p), 1 \le j \le m)$ can be written as a finite linear combination of elements of $\pi(U_{r_1 \dots r_p})$ with coefficients in the field K.

Now, let us prove that the set $\pi(U_{r_1 \dots r_p})$ is linearly independent over K. Suppose that $\sum_{i=1}^{k} a_i \pi(u_i) = 0$ for some $u_1, \dots, u_k \in U_{r_1 \dots r_p}, a_1, \dots, a_k \in K$. Then the element $h = \sum_{i=1}^{k} a_i u_i \in N$ is reduced with respect to G. Indeed, if a term $u = \theta e_j$ appears in h (so that $u = u_i$ for some $i = 1, \dots, k$), then either u is not a multiple of any $u_{g_\nu}^{(1)}$ or $u = \theta u_{g_\nu}^{(1)}$ for some $\theta \in \Theta$, $1 \le \nu \le d$, such that $\mathrm{ord}_\mu(\theta u_{g_\nu}^{(\mu)}) > r_\mu \ge \mathrm{ord}_\mu u_h^{(\mu)}$ for some μ, $2 \le \mu \le p$. By Proposition 3.7, $h = 0$, whence $a_1 = \cdots = a_k = 0$. This completes the proof of the theorem. □

Theorem 4.1 leads to the following existence theorem which is the central result of this section.

Theorem 4.2. *Let D be the ring of Ore polynomials in the set of variables $X = \{x_1, \dots, x_n\}$ over a field K and let a partition (2.1) of the set X be fixed. (As before, $n_i = \mathrm{Card}\, X_i$ for $i = 1, \dots, p$ and D is considered together with the standard p-dimensional filtration.) Furthermore, let*

$\{M_{r_1\dots r_p}|(r_1,\dots,r_p)\in\mathbf{Z}^p\}$ be an excellent p-dimensional filtration of a D-module M. Then there exists a polynomial $\phi(t_1,\dots,t_p)\in\mathbf{Q}[t_1,\dots,t_p]$ such that

(i) *$\phi(r_1,\dots,r_p)=\dim_K M_{r_1\dots r_p}$ for all sufficiently large $(r_1,\dots,r_p)\in\mathbf{Z}^p$;*
(ii) *$\deg_{t_i}\phi\le n_i$ $(1\le i\le p)$, so that $\deg\phi\le n$ and the polynomial ϕ can be represented as $\phi(t_1,\dots,t_p)=\sum_{i_1=0}^{n_1}\cdots\sum_{i_p=0}^{n_p}a_{i_1\dots i_p}\binom{t_1+i_1}{i_1}\dots\binom{t_p+i_p}{i_p}$ where $a_{i_1\dots i_p}\in\mathbf{Z}$ for all $i_1,\dots,i_p$.*

Proof. Since the p-dimensional filtration of M is excellent, there exists an element $(h_1,\dots,h_p)\in\mathbf{Z}^p$ such that $D_{r_1,\dots,r_p}M_{h_1,\dots,h_p}=M_{r_1+h_1,\dots,r_p+h_p}$ for all $(r_1,\dots,r_p)\in\mathbf{N}^p$. It follows that $M=\sum_{i=1}^m Dy_i$ for some elements $y_1,\dots,y_m\in M_{h_1,\dots,h_p}$. Let E be a free D-module with a basis $e_1,\dots,e_m$, let N be the kernel of the natural epimorphism $\pi:E\to M$ ($\pi(e_i)=y_i$ for $i=1,\dots,m$), and let the set $U_{r_1\dots r_p}$ ($r_1,\dots,r_p\in\mathbf{N}$) be the same as in the conditions of Theorem 4.1. Furthermore, let $G=\{g_1,\dots,g_d\}$ be a Gröbner basis of N with respect to $<_1,\dots,<_p$. By Theorem 4.1, for any $r_1,\dots,r_p\in\mathbf{N}$, $\pi(U_{r_1,\dots,r_p})$ is a basis of the vector K-space $M_{r_1,\dots,r_p}$. Therefore, $\dim_K M_{r_1,\dots,r_p}=\operatorname{Card}\pi(U_{r_1,\dots,r_p})=\operatorname{Card}U_{r_1,\dots,r_p}$. (It was shown in the second part of the proof of Theorem 4.1 that the restriction of the mapping π on $U_{r_1,\dots,r_p}$ is bijective).

Let $U'_{r_1,\dots,r_p}=\{w\in U_{r_1,\dots,r_p}|w$ is not a multiple of any element $u_{g_i}^{(1)}$ $(1\le i\le d)\}$ and let $U''_{r_1,\dots,r_p}=\{w\in U_{r_1,\dots,r_p}|$ there exists $g_j\in G$ and $\theta\in\Theta$ such that $w=\theta u_{g_j}^{(1)}$ and $\operatorname{ord}_\nu(\theta u_{g_j}^{(\nu)})>r_\nu$ for some $\nu, 2\le\nu\le p\}$. Then $U_{r_1,\dots,r_p}=U'_{r_1,\dots,r_p}\bigcup U''_{r_1,\dots,r_p}$ and $U'_{r_1,\dots,r_p}\bigcap U''_{r_1,\dots,r_p}=\emptyset$, whence $\operatorname{Card}U_{r_1,\dots,r_p}=\operatorname{Card}U'_{r_1,\dots,r_p}+\operatorname{Card}U''_{r_1,\dots,r_p}$.

By Theorem 2.5, there exists a numerical polynomial $\omega(t_1,\dots,t_p)$ in p variables $t_1,\dots,t_p$ such that $\omega(r_1,\dots,r_p)=\operatorname{Card}U'_{r_1,\dots,r_p}$ for all sufficiently large $(r_1,\dots,r_p)\in\mathbf{N}^p$. In order to express $\operatorname{Card}U''_{r_1,\dots,r_p}$ in terms of $r_1,\dots,r_p$, let us set $a_{ij}=\operatorname{ord}_i u_{g_j}^{(1)}$ and $b_{ij}=\operatorname{ord}_i u_{g_j}^{(i)}$ for $i=1,\dots,p$; $j=1,\dots,d$. Clearly, $a_{1j}=b_{1j}$ and $a_{ij}\le b_{ij}$ for $i=1,\dots,p$; $j=1,\dots,d$. Furthermore, for any $\mu=1,\dots,p$ and for any integers $k_1,\dots,k_\mu$ such that $2\le k_1<\dots<k_\mu\le p$, let $V_{j;k_1,\dots,k_\mu}(r_1,\dots,r_p)=\{\theta u_{g_j}^{(1)}|\operatorname{ord}_i\theta\le r_i-a_{ij}$ for $i=1,\dots,p$ and $\operatorname{ord}_\nu\theta>r_\nu-b_{\nu j}$ if and only if ν is equal to one of the numbers $k_1,\dots,k_\mu\}$.

Then $\operatorname{Card}V_{j;k_1,\dots,k_\mu}(r_1,\dots,r_p)=\phi_{j;k_1,\dots,k_\mu}(r_1,\dots,r_p)$, where

$$\begin{aligned}\phi_{j;k_1,\dots,k_\mu}(t_1,\dots,t_p)={}&\binom{t_1+n_1-b_{1j}}{n_1}\dots\binom{t_{k_1-1}+n_{k_1-1}-b_{k_1-1,j}}{n_{k_1-1}}\\ &\left[\binom{t_{k_1}+n_{k_1}-a_{k_1,j}}{n_{k_1}}-\binom{t_{k_1}+n_{k_1}-b_{k_1,j}}{n_{k_1}}\right]\binom{t_{k_1+1}+n_{k_1+1}-b_{k_1+1,j}}{n_{k_1+1}}\\ &\dots\binom{t_{k_\mu-1}+n_{k_\mu-1}-b_{k_\mu-1,j}}{n_{k_\mu-1}}\left[\binom{t_{k_\mu}+n_{k_\mu}-a_{k_\mu,j}}{n_{k_\mu}}-\binom{t_{k_\mu}+n_{k_\mu}-b_{k_\mu,j}}{n_{k_\mu}}\right]\\ &\dots\binom{t_p+n_p-b_{pj}}{n_p}.\end{aligned}\tag{4.1}$$

(By Theorem 2.5(iii), $\operatorname{Card}\{\theta\in\Theta|\operatorname{ord}_1\theta\le r_1,\dots,\operatorname{ord}_p\theta\le r_p\}=\prod_{i=1}^p\binom{r_i+n_i}{n_i}$ for any $r_1,\dots,r_p\in\mathbf{N}$.) Clearly, $\deg_{t_i}\phi_{j;k_1,\dots,k_\mu}\le n_i$ for $i=1,\dots,p$.

Now, for any $j=1,\dots,d$, let $V_j(r_1,\dots,r_p)=\{\theta u_{g_j}^{(1)}|\operatorname{ord}_i\theta\le r_i-a_{ij}$ for $i=1,\dots,p$ and there exists $\nu\in\mathbf{N}$, $2\le\nu\le p$, such that $\operatorname{ord}_\nu\theta>r_\nu-b_{\nu j}\}$. Then the combinatorial principle of inclusion and exclusion implies that Card $V_j(r_1, \ldots, r_p) = \phi_j(r_1, \ldots, r_p)$, where

$$\phi_j(t_1, \ldots, t_p) = \sum_{k_1=1}^{p} \phi_{j;k_1}(t_1, \ldots, t_p) - \sum_{1 \le k_1 < k_2 \le p} \phi_{j;k_1,k_2}(t_1, \ldots, t_p) + \cdots + (-1)^{\mu-1} \sum_{1 \le k_1 < \cdots < k_\mu \le p} \phi_{j;k_1,\ldots,k_\mu}(t_1, \ldots, t_p) + \cdots + (-1)^{p-1}\phi_{j;2,\ldots,p}(t_1, \ldots, t_p).$$

Obviously, $\deg_{t_i} \phi_j(t_1, \ldots, t_p) \le n_i$ for $i = 1, \ldots, p$. Applying the principle of inclusion and exclusion once again we obtain that Card $U''_{r_1 \ldots r_p} =$ Card $\bigcup_{j=1}^{d} V_j(r_1, \ldots, r_p) = \sum_{j=1}^{d}$ Card $V_j(r_1, \ldots, r_p) - \sum_{1 \le j_1 < j_2 \le d}$ Card $(V_{j_1}(r_1, \ldots, r_p) \bigcap V_{j_2}(r_1, \ldots, r_p)) + \cdots + (-1)^{d-1}$Card $\bigcap_{\nu=1}^{d} V_{j_\nu}(r_1, \ldots, r_p)$, so it remains to prove that if $1 \le j_1 < \cdots < j_s \le d$, then Card $(V_{j_1}(r_1, \ldots, r_p) \bigcap \cdots \bigcap V_{j_s}(r_1, \ldots, r_p)) = \phi_{j_1,\ldots,j_s}(r_1, \ldots, r_p)$, where $\phi_{j_1,\ldots,j_s}(t_1, \ldots, t_p)$ is a numerical polynomial in p variables $t_1, \ldots, t_p$ such that $\deg_{t_i} \phi_{j_1,\ldots,j_s} \le n_i$ for $i = 1, \ldots, p$. It is clear that the intersection $V_{j_1}(r_1, \ldots, r_p) \bigcap \cdots \bigcap V_{j_s}(r_1, \ldots, r_p)$ is not empty (therefore, $\phi_{j_1,\ldots,j_s} \ne 0$) if and only if the leaders $u^{(1)}_{g_{j_1}}, \ldots, u^{(1)}_{g_{j_s}}$ contain the same indeterminate e_i ($1 \le i \le m$). Let us consider such an intersection $V_{j_1}(r_1, \ldots, r_p) \bigcap \cdots \bigcap V_{j_s}(r_1, \ldots, r_p)$, let $v(j_1, \ldots, j_s) = lcm(u^{(1)}_{j_1}, \ldots, u^{(1)}_{j_s})$, and let $v(j_1, \ldots, j_s) = \gamma_\nu u_{g_{j_\nu}}$ ($1 \le \nu \le s; \gamma_\nu \in \Theta$). Then $V_{j_1}(r_1, \ldots, r_p) \bigcap \cdots \bigcap V_{j_s}(r_1, \ldots, r_p)$ is the set of all terms $u = \theta v(j_1, \ldots, j_s)$ such that $\mathrm{ord}_i u \le r_i$ (that is, $\mathrm{ord}_i \theta \le r_i - \mathrm{ord}_i v(j_1, \ldots, j_s)$) for $i = 1, \ldots, p$, and for any $l = 1, \ldots, s$, there exists at least one index $\nu \in \{2, \ldots, p\}$ such that $\mathrm{ord}_\nu(\theta\gamma_l u^{(\nu)}_{g_{j_l}}) > r_\nu$ (i.e., $\mathrm{ord}_\nu\theta > r_\nu - \mathrm{ord}_\nu v(j_1, \ldots, j_s) - \mathrm{ord}_\nu u^{(\nu)}_{g_{j_l}} + \mathrm{ord}_\nu u^{(1)}_{g_{j_l}}$). Denoting $\mathrm{ord}_i v(j_1, \ldots, j_s)$ by $c^{(i)}_{j_1,\ldots,j_s}$ ($1 \le i \le p$) and applying the principle of inclusion and exclusion one more time, we obtain that Card $\bigcap_{\mu=1}^{s} V_{j_\mu}(r_1, \ldots, r_p)$ is an alternating sum of terms of the form Card $W(j_1, \ldots, j_s; k_{11}, k_{12}, \ldots, k_{1q_1}, k_{21}, \ldots, k_{sq_s}; r_1, \ldots, r_p)$ where $W(j_1, \ldots, j_s; k_{11}, k_{12}, \ldots, k_{1q_1}, k_{21}, \ldots, k_{sq_s}; r_1, \ldots, r_p) = \{\theta \in \Theta | \mathrm{ord}_i\theta \le r_i - c^{(i)}_{j_1,\ldots,j_s}$ for $i = 1, \ldots, p$, and for any $l = 1, \ldots, s$, $\mathrm{ord}_k\theta > r_k - c^{(k)}_{j_1,\ldots,j_s} + a_{kj_l} - b_{kj_l}$ if and only if $k = k_{li}$ for some $i = 1, \ldots, q_l\}$ ($q_1, \ldots, q_s$ are some positive integers from the set $\{1, \ldots, p\}$ and $\{k_{i\mu} | 1 \le i \le s, 1 \le \mu \le q_s\}$ is a family of integers such that $2 \le k_{i1} < k_{i2} < \cdots < k_{iq_i} \le p$ for $i = 1, \ldots, s$). Thus, it is sufficient to show that Card $W(j_1, \ldots, j_s; k_{11}, \ldots, k_{sq_s}; r_1, \ldots, r_p) = \psi^{j_1,\ldots,j_s}_{k_{11},\ldots,k_{sq_s}}(r_1, \ldots, r_p)$ where $\psi^{j_1,\ldots,j_s}_{k_{11},\ldots,k_{sq_s}}(t_1, \ldots, t_p)$ is a numerical polynomial in p variables $t_1, \ldots, t_p$ such that $\deg_{t_i} \psi^{j_1,\ldots,j_s}_{k_{11},\ldots,k_{sq_s}} \le n_i$ ($i = 1, \ldots, p$). But this is almost evident: as in the evaluation of Card $V_{j;k_1,\ldots,k_q}(r_1, \ldots, r_p)$ (when we used Theorem 2.5(iii) to obtain formula (4.1)), the number of elements of the set $W(j_1, \ldots, j_s; k_{11}, \ldots, k_{sq_s}; r_1, \ldots, r_p)$ is a product of terms of the form $\binom{r_\nu + n_\nu - c^{(\nu)}_{j_1,\ldots,j_s} - S_\nu}{n_\nu}$ (such a term corresponds to a number $\nu \in \{1, \ldots, p\}$ that is different from all $k_{i\mu}$ ($1 \le i \le s, 1 \le \mu \le q_s$); S_ν is defined as $\max\{b_{\nu j_l} - a_{\nu j_l} | 1 \le l \le s\}$) and terms of the form $[\binom{r_\nu + n_\nu - c^{(\nu)}_{j_1,\ldots,j_s}}{n_\nu} - \binom{r_\nu + n_\nu - c^{(\nu)}_{j_1,\ldots,j_s} - S'_\nu}{n_\nu}]$ (such a term appears in the product if $\nu = k_{i\mu}$ for some i, μ; if $k_{i_1\mu_1}, \ldots, k_{i_e\mu_e}$ are all elements of the set $\{k_{i\mu} | 1 \le i \le s, 1 \le \mu \le q_s\}$ that are equal to ν ($1 \le e \le s, 1 \le i_1 < \cdots < i_e \le s$), then S'_ν is defined as $\min\{b_{\nu j_{i_\lambda}} - a_{\nu j_{i_\lambda}} | 1 \le \lambda \le l\}$). The corresponding numerical polynomial $\psi^{j_1,\ldots,j_s}_{k_{11},\ldots,k_{sq_s}}(t_1, \ldots, t_p)$ is a product of p factors, each of which is equal to either $\binom{t_\nu + n_\nu - c^{(\nu)}_{j_1,\ldots,j_s} - S_\nu}{n_\nu}$ or $[\binom{t_\nu + n_\nu - c^{(\nu)}_{j_1,\ldots,j_s}}{n_\nu} - \binom{t_\nu + n_\nu - c^{(\nu)}_{j_1,\ldots,j_s} - S'_\nu}{n_\nu}]$.

Since the degree of such a product with respect to any t_i does not exceed n_i, this completes the proof. $\square$

Definition 4.3. The polynomial $\phi(t_1, \dots, t_p)$, whose existence is established by Theorem 4.2, is called a dimension polynomial of the D-module M associated with the p-dimensional filtration $\{M_{r_1 \dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$.

Example 4.4. With the above notation, let $n = 2$, $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, and let a D-module M be generated by one element z that satisfies the defining equation $x_1 z + x_2^2 z + z = 0$. In other words, M is a factor module of a free D-module $E = De$ with a free generator e by its D-submodule $N = D(x_1 + x_2^2 + 1)e$. It is easy to see that the set consisting of a single element $g = (x_1 + x_2^2 + 1)e$ is a Gröbner basis of N with respects to the orders $<_1$, $<_2$. In this case, the proof of Theorem 4.2 shows that the dimension polynomial of M associated with the natural bifiltration $\left(M_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} K x_1^i x_2^j z\right)_{r,s \in \mathbf{N}}$ is as follows:

$$\begin{aligned} \phi(t_1, t_2) &= \left[\binom{t_1+1}{1}\binom{t_2+1}{1} - \binom{t_1}{1}\binom{t_2+1}{1}\right] + \binom{t_1}{1}\left[\binom{t_2+1}{1} - \binom{t_2-1}{1}\right] \\ &= 2t_1 + t_2 + 1. \end{aligned}$$

(With the notation of the proof of Theorem 4.2, the polynomial in the first brackets gives Card U'_{rs} while the polynomial $\binom{t_1}{1}[\binom{t_2+1}{1} - \binom{t_2-1}{1}]$ gives Card U''_{rs} for all sufficiently large $(r, s) \in \mathbf{N}^2$.)

As we have seen, every finite set of generators $\{f_1, \dots, f_m\}$ of a finitely generated D-module M produces an excellent p-dimensional filtration of M, and therefore a dimension polynomial of M associated with this filtration. Generally speaking, different finite systems of generators of M produce different dimension polynomials, however every dimension polynomial carries certain integers that do not depend on generators it is associated with. These integers, that characterize the D-module M, are called *invariants* of a dimension polynomial. In what follows, we describe some of such invariants.

For any permutation $(j_1, \dots, j_p)$ of the set $\{1, \dots, p\}$, let $\le_{j_1,\dots,j_p}$ denote a lexicographic order on $\mathbf{N}^p$ defined as follows: $(r_1, \dots, r_p) \le_{j_1,\dots,j_p} (s_1, \dots, s_p)$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbf{N}$, $1 \le k \le p-1$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$. Furthermore, for any set $\Sigma \subseteq \mathbf{N}^p$, let Σ' denote the set $\{e \in \Sigma | e$ is a maximal element of Σ with respect to one of the $p!$ lexicographic orders $\le_{j_1,\dots,j_p}\}$. For example, if $\Sigma = \{(3,0,2), (2,1,1), (0,1,4), (1,0,3), (1,1,6), (3,1,0), (1,2,0)\} \subseteq \mathbf{N}^3$, then $\Sigma' = \{(3,0,2), (3,1,0), (1,1,6), (1,2,0)\}$.

The following result is a consequence of the fact that if $\{M_{r_1\dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ and $\{M'_{r_1\dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ are two excellent p-dimensional filtrations of the same finitely generated D-module M, then there exists an element $(s_1, \dots, s_p) \in \mathbf{N}^p$ such that $M_{r_1\dots r_p} \subseteq M'_{r_1+s_1,\dots,r_p+s_p}$ and $M'_{r_1\dots r_p} \subseteq M_{r_1+s_1,\dots,r_p+s_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$.

Theorem 4.5. *With the above notation, let M be a finitely generated D-module and $\phi(t_1, \dots, t_p) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} a_{i_1\dots i_p} \binom{t_1+i_1}{i_1} \dots \binom{t_p+i_p}{i_p}$ the dimension polynomial associated with some excellent p-dimensional filtration of M. Let $\Sigma_\phi = \{(i_1, \dots, i_p) \in \mathbf{N}^p \,|\, 0 \le i_k \le n_k$ $(k = 1, \dots, p)$ and $a_{i_1\dots i_p} \ne 0\}$. Then $d = \deg \phi$, $a_{n_1\dots n_p}$, the elements $(k_1, \dots, k_p) \in \Sigma'_\phi$, the corresponding coefficients $a_{k_1\dots k_p}$, and the coefficients of the terms of total degree d do not depend on the choice of the excellent filtration.* $\square$

We conclude with the result that generalizes the Kolchin theorem on differential dimension polynomial (see Theorem 2.9).

From now on, let K denote a differential field of zero characteristic with a basic set of derivation operators $\Delta = \{\delta_1, \dots, \delta_m\}$ (as in Section 2, K will be also called a Δ-field). Let us fix a partition of the set Δ into p disjoint finite sets ($p \geq 1$): $\Delta = \Delta_1 \bigcup \dots \bigcup \Delta_p$, where $\Delta_1 = \{\delta_1, \dots, \delta_{m_1}\}, \dots, \Delta_p = \{\delta_{m_1+\dots+m_{p-1}+1}, \dots, \delta_m\}$. Furthermore, let Θ_Δ denote the free commutative semigroup generated by the set Δ and for any element $\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta_\Delta$, let $\mathrm{ord}_i\, \theta$ denote the order of θ with respect to Δ_i, that is $\mathrm{ord}_i\, \theta = \sum_{j=1}^{m_i} k_{m_1+\dots+m_{i-1}+j}$ ($1 \leq i \leq p$) and $\mathrm{ord}_1 \theta = \sum_{j=1}^{m_1} k_j$. Finally, for any non-negative integers $r_1, \dots, r_p$, let $\Theta_\Delta(r_1, \dots, r_p) = \{\theta \in \Theta_\Delta \mid \mathrm{ord}_i\, \theta \leq r_i \text{ for } i = 1, \dots, p\}$.

Theorem 4.6. *With the above notation, let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a polynomial $\Phi_\eta(t_1, \dots, t_p) \in \mathbf{Q}[t_1, \dots, t_p]$ such that*

(i) $\Phi_\eta(r_1, \dots, r_p) = \mathrm{trdeg}_K K(\{\theta\eta_j \mid \theta \in \Theta_\Delta(r_1, \dots, r_p), 1 \leq j \leq n\})$ *for all sufficiently large* $(r_1, \dots, r_p) \in \mathbf{N}^p$.

(ii) $\deg_{t_i} \Phi_\eta \leq m_i$ $(1 \leq i \leq p)$*, so that* $\deg \Phi_\eta \leq m$ *and the polynomial* $\Phi_\eta(t_1, \dots, t_p)$ *can be represented as* $\Phi_\eta = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1+i_1}{i_1} \dots \binom{t_p+i_p}{i_p}$ *where* $a_{i_1 \dots i_p} \in \mathbf{Z}$ *for all* $i_1, \dots, i_p$.

Proof. Let $\mathrm{Der}_K L$ denote the vector L-space of all K-linear derivations of the field L (i.e., the set of all derivations $D : L \longrightarrow L$ such that $D(a) = 0$ for any $a \in K$), and let $\Omega_K(L)$ be the module of differentials of the extension L/K (recall that $\Omega_K(L)$ is a subspace of the vector L-space $(\mathrm{Der}_K L)^* = Hom_L(\mathrm{Der}_K L, L)$ generated by all mappings $d\eta$ ($\eta \in L$) such that $d\eta(\delta) = \delta(\eta)$ for any $\delta \in \mathrm{Der}_K L$). Let R denote the ring of differential operators over L (see Example 2.1.1). As it was shown in Johnson (1969a), $\Omega_K(L)$ can be considered as an R-module such that $\gamma(d\eta) = d\gamma(\eta)$ for any $\gamma \in \Delta$, $\eta \in L$. Furthermore, the elements $d\eta_i$ ($1 \leq i \leq n$) generate $\Omega_K(L)$ as an R-module and a set $\Sigma \subseteq L$ is algebraically independent over K if and only if the set $\{d\xi \mid \xi \in \Sigma\}$ is linearly independent over L. It follows that if $\Omega_K(L)_{r_1 \dots r_p}$ ($r_1, \dots, r_p \in \mathbf{N}$) is a vector L-subspace of $\Omega_K(L)$ generated by the set $\{d\eta \mid \eta \in K(\{\theta\eta_j \mid \theta \in \Theta_\Delta(r_1, \dots, r_p), 1 \leq j \leq n\})$ and $\Omega_K(L)_{r_1 \dots r_p} = 0$, if at least one of the numbers $r_1, \dots, r_p$ is negative, then $(\Omega_K(L)_{r_1 \dots r_p})_{r_1, \dots, r_p \in \mathbf{Z}}$ is an excellent p-dimensional filtration of the R-module $\Omega_K(L)$ (R is treated as a ring of Ore polynomials over L, see Example 2.1.1). Furthermore, $\dim_L \Omega_K(L)_{r_1 \dots r_p} = \mathrm{tr}\deg_K K(\{\theta\eta_j \mid \theta \in \Theta_\Delta(r_1, \dots, r_p), 1 \leq j \leq n\})$ for all $r_1, \dots, r_p \in \mathbf{N}$. Now, applying Theorem 4.2 we obtain the statements of our theorem. □

Definition 4.7. Numerical polynomial $\Phi_\eta(t_1, \dots, t_p)$, whose existence is established by Theorem 4.6, is called the $(\Delta_1, \dots, \Delta_p)$-dimension polynomial of the Δ-field extension L/K associated with the set of differential generators $\eta = \{\eta_1, \dots, \eta_n\}$.

As a consequence of Theorem 4.5 we obtain the following result that describes invariants carried by a $(\Delta_1, \dots, \Delta_p)$-dimension polynomial of a differential field extension.

Theorem 4.8. *With the notation of Theorem 4.6, let $\Phi_\eta(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1+i_1}{i_1} \dots \binom{t_p+i_p}{i_p}$ be the $(\Delta_1, \dots, \Delta_p)$-dimension polynomial of a difference field extension $L \supseteq K$ associated with a set of differential generators $\eta = \{\eta_1, \dots, \eta_n\}$. Furthermore, let $E_\eta = \{(i_1, \dots, i_p) \in \mathbf{N}^p \mid 0 \leq i_k \leq m_k \ (k = 1, \dots, p) \text{ and } a_{i_1 \dots i_p} \neq 0\}$. Then*

$d = \deg \Phi_\eta$, $a_{m_1 \dots m_p}$ the elements $(k_1, \dots, k_p) \in E'_\eta$, the corresponding coefficients $a_{k_1 \dots k_p}$, and the coefficients of the terms of total degree d do not depend on the choice of the system of Δ-generators η. $\square$

Example 4.9. Let K be a differential field with a basic set of derivation operators $\Delta = \{\delta_1, \delta_2, \delta_3\}$ and let L be a Δ-field extension of K generated by a single Δ-generator η with the defining equation

$$\delta_1^a \delta_2^b \delta_3^c \eta + \delta_3^{a+b+c} \eta = 0 \tag{4.2}$$

where a, b and c are some positive integers. In other words, $L = K\langle\eta\rangle$ is Δ-isomorphic to the quotient field of the factor ring $K\{y\}/P$ where P is the linear Δ-ideal of the ring of differential (Δ-) polynomials $K\{y\}$ generated by the Δ-polynomial $\delta_1^a \delta_2^b \delta_3^c y + \delta_3^{a+b+c} y$. By Kondrateva et al. (1999, Example 5.7.5), the classical Kolchin differential dimension polynomial $\omega_{\eta/K}(t)$ of the extension $K\rangle\eta\langle/K$ is as follows: $\omega_{\eta/K}(t) = \binom{t+3}{3} - \binom{t+3-(a+b+c)}{3} = (\frac{a+b+c}{2})t^2 + (\frac{(a+b+c)(4-a-b-c)}{2})t + \frac{(a+b+c)[(a+b+c)^2-6(a+b+c)+11]}{6}$.

Now, let us fix a partition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$ with $\Delta_i = \{\delta_i\}$ $(i = 1, 2, 3)$ and let $\Phi_\eta(t_1, t_2, t_3)$ denote the $(\Delta_1, \Delta_2, \Delta_3)$-dimension polynomial of L/K associated with this partition and the Δ-generator η. Furthermore, let D denote the ring of differential (Δ-) operators over L. It follows from the proof of Theorem 4.6 that Φ_η coincides with the dimension polynomial of the D-module $\Omega_K(L)$ associated with the excellent 3-dimensional filtration defined by the generator $d\eta$. As it is shown in the proof of Theorem 4.2, the last polynomial is the sum of two polynomials that describe the size of the sets $U'_{r_1r_2r_3}$ and $U''_{r_1r_2r_3}$ (we use the notation of the proof). By Theorem 2.7 (and with its notation), the polynomial that describes the size of the first set is as follows: $\omega_{(a,b,c)}(t_1, t_2, t_3) = \sum_{k=0}^{1} \sum_{\sigma \in \Gamma(k,1)} \binom{t_1+1-b_{\sigma 1}}{1}\binom{t_2+1-b_{\sigma 2}}{1}\binom{t_3+1-b_{\sigma 3}}{1} = ct_1t_2 + b_1t_3 + at_2t_3 + (b + c - bc)t_1 + (a + c - ac)t_2 + (a + b - ab)t_3 + a + b + c - ab - ac - bc + abc$. The direct computation of the second polynomial (as in the proof of Theorem 4.2) gives the expression $\phi(t_1, t_2, t_3) = (a + b + 1)(t_1 - a + 1)(t_2 - b + 1)$ whence $\Phi_\eta(t_1, t_2, t_3) = \omega_{(a,b,c)}(t_1, t_2, t_3) + \phi(t_1, t_2, t_3) = (a+b+c+1)t_1t_2 + bt_1t_2 + at_2t_3 + (a+b+c+1-ab-b^2-bc)t_1 + (a+b+c+1-ab-a^2-ac)t_2 + (a+b-ab)t_3 + a+b+c+1+ab^2+a^2b-a^2-b^2-2ab-bc-ac+abc$.

Comparing this polynomial with $\omega_{\eta/K}(t)$ we see that the Kolchin dimension polynomial carries two differential birational invariants, its degree 1 and the leading coefficient $a + b + c$, while $\Phi_\eta(t_1, t_2, t_3)$, according to Theorem 4.8, carries four such invariants, its total degree 1, $a + b + c + 1$, a, and b. Therefore, the polynomial Φ_η determines all three parameters a, b and c of the defining differential equation (4.2) while $\omega_{\eta/K}(t)$ gives just the sum of the parameters.

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